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RADIÁLISAN GRADIENS ORTOTROP TÁRCSÁK FESZÜLTSÉG- ÉS ELMOZDULÁSMEZŐINEK SZÁMÍTÁSA

DETERMINATION OF STRESSES AND DISPLACEMENT IN ORTHOTROPIC RADIALLY GRADED DISKS

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ABSTRACT

This paper deals with the determination of stress distribution and displacement field of radially graded disks. The material behaviour is linearly elastic and orthotropic, the thin disk is subjected to combined mechanical and thermal loading. The material distribution is arbitrary function of the radial coordinate and the temperature field, while the temperature distribution and the thickness profile of the thin disk depend on the radial coordinate. A system of differential equations is derived, then reformulated to transform the steady-state thermoelastic problem to a system of initial value problems. A stress function is used to solve the axisymmetric problem, then different boundary conditions are investigated. The solution is derived, when the displacement values at the cylindrical boundary surfaces are given (e.g. fixed or prescribed).

1. INTRODUCTION

Functionally graded materials (FGMs) represent a class of advanced engineered materials characterized by a continuous variation in composition, which in turn causes a corresponding gradation in material properties aligned with the functional requirements of the structural component, typically along a single spatial direction. The graded interface between constituent phases ensures a smooth transition from one material to another, thereby imparting enhanced mechanical performance and improved thermal resistance. Owing to these advantageous characteristics, the concept of FGMs has attracted increasing attention in recent decades. FGMs are important in case of inverse design, which is a computational approach where the desired behaviour of a material or structure is specified first, and then the material distribution (or geometry) is determined to achieve that target. Inverse design is particularly valuable in this case, because it allows engineers to tailor the spatial variation of material properties - such as stiffness, thermal behaviour or strength - to meet specific functional requirements.

A substantial body of research has investigated the mechanics of functionally graded materials from diverse perspectives. Comprehensive treatments of linearly elastic problems in non-homogeneous solids can be found in several monographs [1–4]. In addition, numerous studies have reported analytical, semi-analytical, and numerical solutions to thermomechanical problems in a variety of geometries, including hollow spheres, disks, cylindrical shells, beams, such as [5-7]. A substantial body of research has been devoted to the thermoelastic and mechanical analysis of functionally graded and orthotropic disks, employing analytical, semi-analytical, and numerical methodologies. Early contributions include the work of Pen and Li [8], who investigated steady-state thermoelastic problems in isotropic radially graded disks with arbitrary radial non-homogeneity, formulating the solution as a Fredholm integral equation. Stampouloglou and Theotokoglou [9] derived exact solutions for hollow circular cylinders and disks with exponential and power-law variations in shear modulus, based on compatibility and equilibrium equations. Subsequent studies expanded the range of loading and geometric conditions. For instance, the analysis in [10] considered displacement and stress fields in radially graded hollow disks subjected to angular acceleration and thermal effects using a semi-analytical approach. Boğa and Yildirim [11] employed the method of complementary functions to study disks with parabolic thickness profiles, while Gönczi [12] proposed a numerical scheme for the steady-state thermoelastic response of isotropic hollow circular disks with arbitrary radial gradation. Rotating disks have received particular attention. Zheng et al. [13, 14] analysed radially graded isotropic and fiber-reinforced rotating disks, applying finite difference methods to compute displacements and stress distributions. Eraslan et al. [15] developed analytical solutions for orthotropic disks with power-law property profiles, transforming the governing relations into hypergeometric differential equations. More recent works have addressed orthotropic and specially tailored material distributions. Yildirim [16] applied the complementary function method to thermomechanical problems of orthotropic disks, while Allam et al. [17] introduced semi-analytical formulations for specific gradation patterns. Variational approaches on discretized domains were also adopted in [18, 19] to investigate the

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thermoelastic response of orthotropic disks. Collectively, these studies underscore the variety of mathematical tools - ranging from integral equation formulations to complementary function methods, finite difference schemes, and variational techniques - that have been developed to capture the complex behaviour of radially graded and orthotropic disks under diverse mechanical and thermal loading conditions.

The subject of this study is the solution of a thermoelasticity problem for cylindrically orthotropic, radially graded disks. The structural element is subjected to combined thermal and mechanical loading, which can be described by an axisymmetric model. The time dependence is neglected, therefore, the problem can be treated as a decoupled thermoelastic one. Due to the geometry and loading conditions, the disk is in a state of plane stress. The sketch of the axisymmetric disk is shown in Fig. 1. Here, the outer radius is denoted by R_2, the inner radius by R_1, and the thickness profile in the axial (z) direction is represented by $h(r)$, which is an arbitrary function of the radial coordinate.

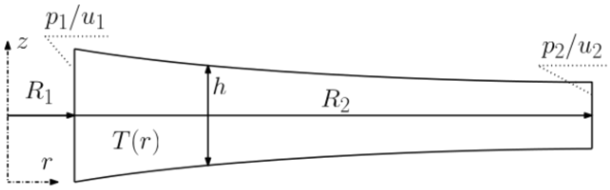

Figure 1. The sketch of the problem

The applied loads on the structural element consist of a temperature field depending arbitrarily on the radial coordinate, as well as a body force acting in the radial direction. In addition, constant radial surface tractions may be applied on the inner and outer curved boundaries (p_1 , p_2), or alternatively, prescribed radial displacements may be imposed on these boundaries (u_1, u_2 as "loading alternatives"). The numerical solution method employed reduces the problem to a combination of three initial-value problems, relying on the linearity of the governing system of equations and boundary conditions. The material properties $C_{ij}(r,T)$ are considered as arbitrary functions of both the radial coordinate and the temperature field.

2. FORMULATION OF THE PROBLEM

Let's consider a cylindrical coordinate system (r,φ,z). The formulated disk problem is treated as a linear problem. Small deformations are assumed, however, due to the nature of the structure and loading, the displacement field has only a radial component. The kinematic equations of the problem are

$$\varepsilon_r(r) = \frac{\partial u_r}{\partial r} = \frac{\mathrm{d}u_r(r)}{\mathrm{dr}}, \tag{1}$$

$$\varepsilon_\varphi(r) = u_r r^{-1}. \tag{2}$$

In our previous equations u_r is the radial displacement, $\varepsilon_r, \varepsilon_\varphi$ denote the radial and circumferential normal strains. In our axisymmetric case, the nonzero equilibrium equation (in the radial direction)

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\varphi}}{\partial\varphi} + \frac{\partial\tau_{rz}}{\partial z} + \frac{1}{r}\left(\sigma_r - \sigma_\varphi\right) + b_r = 0 \tag{3}$$

leads us to

$$\frac{d}{dr}\left[r\sigma_r(r)h(r)\right] - h(r)\sigma_\varphi(r) + h(r)b_r = 0, \tag{4}$$

where the body forces are expressed as the product of the thickness of the disk and a body force function b_r. In case of rotating disk, we have $b_r = \rho(r)\omega^2 r^2$, where ϱ is the density of the material. From Eq. (4) the stress function of the problem can be derived. Let's denote the stress function by S, which can be expressed as

$$\sigma_r(r) = S(r)r^{-1}h^{-1}(r), \tag{5}$$

$$\sigma_\varphi(r) = \frac{dS(r)}{dr}\frac{1}{h(r)} + b_r(r) \tag{6}$$

for our axisymmetric disks. The cylindrically orthotropic thermoelastic constitutive behaviour, the Duhamel-Neumann equation [4] is expressed in Voigt's engineering notation as $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} + \boldsymbol{\beta}\vartheta$, where the 6×6 stiffness matrix $\mathbf{C}$ contains nine independent material parameters. $\vartheta = T - T_{eq}$ is the temperature difference function, where the absolute temperature is compared to the equilibrium temperature T_{eq}. In the governing equations, the temperature distribution is arbitrary, but necessarily a function of the radial coordinate. This assumption is justified by the disk's small thickness and by prescribing identical third-kind boundary conditions on the upper and lower curved surfaces, which yield, to a good approximation of such temperature profile. Considering linearly elastic, orthotropic materials, the nonzero components of the stress tensor in the cylindrical coordinate system are given as follows [4]

$$\begin{aligned}\sigma_r(r) = C_{11}(T,r)\varepsilon_r(r) + C_{12}(T,r)\varepsilon_\varphi(r) +\\ +C_{13}(T,r)\varepsilon_z(r) + \beta_{11}(T,r)\vartheta(r),\end{aligned} \tag{7}$$

$$\begin{aligned}\sigma_\varphi(r) = C_{12}(T,r)\varepsilon_r(r) + C_{22}(T,r)\varepsilon_\varphi(r) +\\ +C_{23}(T,r)\varepsilon_z(r) + \beta_{22}(T,r)\vartheta(r),\end{aligned} \tag{8}$$

furthermore, the plane stress state of the disk leads us to the normal stress

$$\sigma_z = 0 = C_{31}\varepsilon_r + C_{32}\varepsilon_\varphi + C_{33}\varepsilon_z + \beta_{33}\vartheta, \tag{9}$$

$$\varepsilon_z = -C_{31}C_{33}^{-1}\varepsilon_r - C_{33}^{-1}C_{32}\varepsilon_\varphi - \beta_{33}C_{33}^{-1}\vartheta. \tag{10}$$

Let's introduce the following notations for the material properties

$$C_1 = C_{11} - C_{33}^{-1}C_{13}^2, C_2 = C_{12} - C_{33}^{-1}C_{23}C_{13},$$
$$C_3 = C_2 = C_{21} - C_{33}^{-1}C_{31}C_{23}, C_4 = C_{22} - C_{33}^{-1}C_{23}^2, \quad (11)$$
$$B_1 = \beta_{11} - \beta_{33}C_{33}^{-1}C_{13}, B_2 = \beta_{22} - \beta_{33}C_{33}^{-1}C_{23}.$$

Using these new constants we have

$$\sigma_r(r) = C_1(T,r)\varepsilon_r(r) + C_2(T,r)\varepsilon_\varphi(r) + +B_1(T,r)\vartheta(r), \quad (12)$$

$$\sigma_\varphi(r) = C_2(T,r)\varepsilon_r(r) + C_4(T,r)\varepsilon_\varphi(r) + +B_2(T,r)\vartheta(r) \quad (13)$$

for the nonzero normal stress components. From the combination of Eqs. (1), (2), (5) and (12) we have our first equation as

$$\frac{\mathrm{d}u}{\mathrm{d}r} = C_{11}^{-1}r^{-1}h^{-1}S - C_2C_{11}^{-1}r^{-1}u - B_1C_{11}^{-1}\vartheta. \quad (14)$$

From Eqs. (1), (2), (13), (14) and (6) the second differential equation of our system of initial value problem can be determined as

$$\frac{\mathrm{d}S}{\mathrm{d}r} = C_2C_{11}^{-1}r^{-1}S + (C_4 - C_2^2C_{11}^{-1})hr^{-1}u + +(B_2 - B_1C_2^{-1}C_{11}^{-1})h\vartheta - hb_r. \quad (15)$$

To solve the system of differential equations of (14) and (15), we can use different approaches. These techniques solve two initial value problems - for example with Runge-Kutta-Fehlberg method -, from which the actual initial values of the stress function and radial displacement of the original problem can be calculated. When there are traction boundary conditions at both boundaries (R_1, R_2), we can use a similar approach to the one presented in [12]. We can make it more efficient, when for the first calculation we use the initial data ($u_1^1 = 1$, $S_1^1 = 0$) and calculate the values of u_2^1, S_2^1. In our second calculation we use ($u_1^2 = 0$, $S_1^2 = 1$) and determine (u_2^2, S_2^2). From this we can compute the solution of the problem as described in [12] for a similar problem. When we have kinematic boundary conditions $u_r(r = R_1) = u_1$, $u_r(r = R_2) = u_2$, we need the data $u_1^1 = u_1$, $S_1^1 = 1$ in our first initial value problem and calculate the values of u_2^1, S_2^1. In our second calculation we use ($u_1^2 = u_1$, $S_1^2 \neq S_1^1$: arbitrary value) and determine u_2^2, S_2^2. From this calculation the initial value of the stress function is

$$S_1 = (S_1^1 - S_1^2)(u_1 + u_2)(u_2^2 - u_2^1)^{-1} + S_1^1. \quad (16)$$

For the mixed boundary conditions, we can utilize a similar approach, which is based on the linear combination of first two solutions.

3. NUMERICAL EXAMPLE

For the numerical example, the material properties (M) have the following (power-law based) form:

$$M(r) = M_0\left(\frac{r}{R_1}\right)^m,$$

where the data are

$$C_1^0, C_2^0, C_3^0 = 0.45, 0.3, 16 \text{ GPa}; \beta_1^0 = -12500\frac{\mathrm{N}}{\mathrm{m^2K}},$$
$$\beta_2^0 = -32000\frac{\mathrm{N}}{\mathrm{m^2K}}, \lambda_0 = 20\frac{\mathrm{W}}{\mathrm{mK}}, q_0 = 70\frac{W}{m^2K},$$
$$T_{en}(r) = 95 - 3000r^{1.8}[\mathrm{K}], \vartheta_1 = 120\mathrm{K}, \vartheta_2 = 20\mathrm{K},$$
$$R_1 = 0.02\mathrm{m}, R_2 = 0.1\mathrm{m}, h(r) = r^{-0.2}[\mathrm{mm}],$$
$$p_1 = 40\mathrm{MPa}, p_2 = 5\mathrm{MPa}, \mathrm{m} = 12.$$

In the previous expressions λ and q denote the heat transfer and heat exchange coefficients, T_{en} is the environmental temperature along the radial direction. The numerical method to calculate the temperature field is presented in [12] or we can use finite element method to determine it. We are going to validate the equations of the kinematic boundary conditions by calculating the result from the traction boundary conditions (p_1, p_2), then using the computed displacements u_1 and u_2 as kinematic boundary conditions to get the original traction values of p_1, p_2 back. In this example the dominant stress coordinate is the tangential normal stress. Figure 2 shows the relative tangential normal stress distribution ($\sigma_\varphi(r)/p_1$). It is in good agreement with the analytical solution. When we used the displacement values (0.35, 0.1 mm), we got back the radial normal stresses $\sigma_r(r = R_1) = -39.9$ MPa, and $\sigma_r(r = R_2) = -5.05$ MPa.

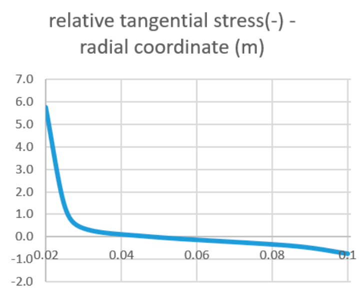

Figure 2. The tangential normal stress distribution

4. SUMMARY

The determination of the displacement field and normal stresses of radially graded orthotropic disks was investigated. Plain stress assumption and Voigt notation was used to formulate the problem. Stress functions were derived and used to transform the problem into a system of initial value problems. Due to the linearity of the problem, the solution can be calculated by solving two initial value problems, then the actual initial values can be calculated and used to solve the thermoelastic problem (e.g. by utilizing Runge-Kutta-Fehlberg method). The presented method works for an arbitrary temperature function and thickness profile, that depends on the radial coordinate, while the material parameters can be arbitrary functions of the radial coordinate and the temperature distribution. A numerical example was considered (that is based on power-law functions), where the solutions of two different boundary conditions were tested on the same problem. The results were in good agreement.

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